1 a) $f(x) = \ln(x) - \frac{x}{3}$ $f'(x) = \frac{1}{x} - \frac{1}{3}$

(1) $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{\ln(2) - 2/3}{1/2 - 1/3} = 1.841117$

$\frac{x_1 - x_0}{x_1} = 0.086297$

(2) $g''(p) = \frac{f''(p)}{f'(p)} = \frac{-1/p^2}{1/p - 1/3} \neq 0 \rightarrow$ no 3^rd order convergence

b) $x_{n+1} = x_n + \alpha\left(\ln(x_n) - \frac{x_n}{3}\right)$

$g'(x) = 1 + \alpha\left(\frac{1}{x} - \frac{1}{3}\right)$ $g'(1.85) = 1 + \alpha\left(\frac{1}{1.85} - \frac{1}{3}\right) = 0$

$\rightarrow \alpha = \frac{-1}{\frac{1}{1.85} - \frac{1}{3}} = -4.826087$

c) $g(x) = e^{x/3}$ $g'(x) = \frac{1}{3} e^{x/3}$

(1) $g'(1.85) = \frac{1}{3} e^{\frac{1.85}{3}} = 0.617581$

$|g'(1.85)| < 1 \rightarrow$ convergence

error reduction:

$\varepsilon_{n+1} \approx 0.617581\, \varepsilon_n$

(2) $K = \frac{x_4 - x_3}{x_3 - x_2} = 0.628678$

$\varepsilon_4 \approx \left| \frac{K}{1-K}(x_4 - x_3) \right| = 0.022718$

(3) $x_4 - \frac{(x_4 - x_3)^2}{(x_4 - x_3) - (x_3 - x_2)} = 1.856616$

2 a) $f(x) = \frac{e^x}{x+1}$ $\quad f'(x) = \frac{e^x}{x+1} - \frac{e^x}{(x+1)^2}$

$$f''(x) = \frac{e^x}{x+1} - 2\frac{e^x}{(x+1)^2} + 2\frac{e^x}{(x+1)^3}$$

(1) for x in $[0,1]$ $f''(x)$ bounded $\longrightarrow$ optimal convergence

(2) $h = 1/2$; $0 \quad \frac{1}{2} \quad 1$

$$\frac{h}{2}\left(f(0) + f(\tfrac{1}{2})\right) + \frac{h}{2}\left(f(\tfrac{1}{2}) + f(1)\right)$$

$$= 0.524787 + 0.614572 = 1.139359$$

(3) $\varepsilon \leq \frac{b-a}{12} h^2 M = \frac{M}{48}$

$$\left.\begin{array}{l} f''(0) = 1-2+2 = 1 \\ f''(1) = e/2 - 2e/4 + 2e/8 = e/4 \\ f''(0.6) = \cdots \approx 0.605 \end{array}\right\} M = 1$$

$\varepsilon \leq \frac{1}{48}$

b) $h = 1$; $0 \quad 1$

(1) $\frac{h}{6}\left(f(0) + 4f(\tfrac{1}{2}) + f(1)\right) = \frac{1}{6}\left(1 + 4\frac{e^{1/2}}{3/2} + \frac{e}{2}\right) = 1.125955$

(2) $\left.\begin{array}{l} M = h f(\tfrac{1}{2}) \\ T = \frac{h}{2}\left(f(0) + f(1)\right) \end{array}\right\} \longrightarrow S = \frac{1}{3}T + \frac{2}{3}M$

leads to $\varepsilon \leq \frac{5}{72} h^2 M$ $<$ 2^{nd} order, using f''

should be 4^{th} order, $f^{(4)}$

c) $q = \frac{I_{64} - I_{128}}{I_{128} - I_{256}} = 4.003861$ $\quad$ 2^{nd} order convergence, as it should be theoretically

(1)

(2) $\varepsilon_{256} \approx \frac{1}{3}|I_{256} - I_{128}| = 8.633333E-7$

$(\frac{1}{4})^n \varepsilon_{256} < 1.0E-8 \longrightarrow n \geq 4 \Longrightarrow 2^4 \cdot 256 = 4096$ segments

(3) $T_2(128) = \frac{4}{3} I_{128} - \frac{1}{3} I_{64} = 1.125386608333$

$T_2(256) = \frac{4}{3} I_{256} - \frac{1}{3} I_{128} = 1.125386608666$

$\varepsilon \approx \frac{1}{15}\left(T_2(256) - T_2(128)\right)$

3 a) $k_1 = \frac{1}{2}(-5+1).2 = -4$

$k_2 = \frac{1}{2}(-4\frac{1}{2}+1).(2+k_1) = 3\frac{1}{2}$

(1) $y(-4\frac{1}{2}) = 2 + \frac{1}{2}(-4+3\frac{1}{2}) = 7/4$

(2) $y_{n+1} = y_n + h(x_{n+1}+1)y_{n+1}$

$y = 2 + \frac{1}{2}(-4\frac{1}{2}+1)y \rightarrow y(-4\frac{1}{2}) = 8/11$

(3) $\{1 - h(x_{n+1}+1)\}y_{n+1} = y_n \rightarrow A = \frac{1}{1-h(x_{n+1}+1)}$

for x in $[-5,-1]$: $1-h(x_{n+1}+1) > 1 \rightarrow 0 < A < 1$ stable

for x in $[-1,0]$: $x=0$ $A = \frac{1}{1-h}$ $1-h<-1$ OR $1-h>1$ never

$\hookrightarrow h>2$

$x=-\frac{1}{2}$ $A = \frac{1}{1-h/2}$ $1-\frac{h}{2}<-1$ OR $1-\frac{h}{2}>1$ never

$\hookrightarrow h>4$

$x=-\frac{9}{10}$ $A=\frac{1}{1-h/10}$ $1-\frac{h}{10}<-1$ OR $1-\frac{h}{10}>1$ never

$\hookrightarrow h>20$

etc $\rightarrow$ never stable

b) (1) decrease x in $[-5,-1]$, then increase in $[-1,0]$

seems to wiggle (although true solution shows same behaviour)

Heun: $-2 < ah < 0 \rightarrow -2 < (x+1)h < 0$

unstable for x in $[-1,0]$

(2) $\varepsilon_{-2} = \frac{1}{3}(y_{320}(-2) - y_{160}(-2)) = 3.15E-6$ $\quad q = 4.457884$

$\varepsilon_{-1} = \frac{1}{3}(y_{320}(-1) - y_{160}(-1)) = 1.9169E-6$ $\quad q = 4.455510$

solution at $x=-1$ smaller, hence err est smaller value

error accumulation holds for one single grid

(3) $(\frac{1}{4})^n < 3.15E-6$ $n \geq 5$ $N = 2^5 . 320 = 10240$

better use RK4, same accuracy with much smaller N

c) $\int_{x_0}^{x_2} y' = \int_{x_0}^{x_2} f(x,y) \Rightarrow y_2 - y_0 = 2h\, f(x_1, y_1)$

$\Rightarrow y_2 - 2 = \frac{2}{4}(-4.75+1)(0.75)$

$\Rightarrow y_2 = 0{,}59375 \quad (= \frac{19}{32})$

4 a)

w1 w2 w3

0 2 4

$h=2$

bound cond : $w_1 = w_3 = 0$

$$\frac{w_3 - 2w_2 + w_1}{(2)^2} = 0.5\, w_2 + 5E\text{-}3(2-4)2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1/4 & -1 & 1/4 & -\frac{2}{100} \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \text{solution} \begin{pmatrix} 0 \\ 2/100 \\ 0 \end{pmatrix} \leftarrow \text{at middle}$$

b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1/4 & -1 & 1/4 & -\frac{2}{100} \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -3/4 & 0 & 0 & -\frac{2}{100} \\ 0 & 0 & 1 & 0 \end{array}\right] \quad w_1 = \frac{8}{300}$$

accuracy w' is $O(\Delta x)$, only first order, accuracy w'' is $O(\Delta x^2)$

$\hookrightarrow$ finer grid needed

c)

$\frac{4}{3}$ $\frac{8}{3}$

L R

0 $\frac{4}{3}$ 4

$$w'_R = \frac{w_3 - w_2}{8/3} \qquad w'_L = \frac{w_2 - w_1}{4/3}$$

$$w''\left(\tfrac{4}{3}\right) = \frac{\frac{w_3 - w_2}{8/3} - \frac{w_2 - w_1}{4/3}}{\frac{2}{3} + \frac{4}{3}}$$

$$= \frac{3}{16} w_3 - \frac{9}{16} w_2 + \frac{3}{8} w_1$$

Programs

1 d) initialisation 2 values
loop construction
Secant iteration formula
shift values for efficiency (only 1 function call per iteration)
error estimate $\left|\frac{x_{n+1} - x_n}{x_{n+1}}\right| < 1E\text{-}6$ stop criterion

2 c) grid between [0,1]
loop construction, summation areas per grid
midpoint formula
refinement procedure, loop over grids
err est $\frac{1}{3}|I_{N/2} - I| < 1E\text{-}6$ stop criterion

3 d) grid between [-5,0]
loop construction, march over grid
RK2/Heun formulas to go from $x_n \to x_{n+1}$
refinement procedure, loop over grids
err est (whole grid, max) $\frac{1}{3}|y_{\frac{N}{2}} - y_N| < 1E\text{-}6$
stop criterion
max